Weakly Nonlinear Density-Velocity Relation

Michał J. CHODOROWSKI[1]

and

Ewa L. ŁOKAS[2]

Copernicus Astronomical Center

Bartycka 18, 00–716 Warsaw, Poland

[1] e-mail: michal@camk.edu.pl

[2] e-mail: lokas@camk.edu.pl

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Abstract

We rigorously derive weakly nonlinear relation between cosmic density and velocity fields up to third order in perturbation theory. The density field is described by the mass density contrast, δ. The velocity field is described by the variable θ proportional to the velocity divergence, $\theta = -f(\Omega)^{-1}H_0^{-1}\nabla \cdot \mathbf{v}$, where $f(\Omega) \simeq \Omega^{0.6}$, Ω is the cosmological density parameter and H_0 is the Hubble constant. Our calculations show that mean δ given θ is a third order polynomial in θ, $\langle\delta\rangle|_\theta = a_1\theta + a_2(\theta^2 - \sigma_\theta^2) + a_3\theta^3$. This result constitutes an extension of the formula $\langle\delta\rangle|_\theta = \theta + a_2(\theta^2 - \sigma_\theta^2)$, found by Bernardeau (1992) which involved second order perturbative solutions. Third order perturbative corrections introduce the cubic term. They also, however, cause the coefficient a_1 to depart from unity, in contrast with the linear theory prediction. We compute the values of the coefficients a_p for scale-free power spectra, as well as for standard CDM, for Gaussian smoothing. The coefficients obey a hierarchy $a_3 \ll a_2 \ll a_1$, meaning that the perturbative series converges very fast. Their dependence on Ω is expected to be very weak. The values of the coefficients for CDM spectrum are in qualitative agreement with the results of N-body simulations by Ganon et al. (1996). The results provide a method for breaking the Ω-bias degeneracy in comparisons of cosmic density and velocity fields such as IRAS-POTENT.

Key Words: cosmology: theory – galaxies: clustering – galaxies: formation – large–scale structure of the Universe

1 Introduction

The most common assumption in theory of structure formation is the gravitational instability hypothesis: the observed large-scale structure has formed by the gravitational amplification of small-amplitude fluctuations present in the primordial density field. Cosmic velocity fields of galaxies result consequently from the gravitational attraction of large-scale mass inhomogeneities, that perturb the uniform Hubble flow. The quantitative relation between the peculiar velocity field, $\mathbf{v}$, and the mass density contrast field, $\delta = \rho/\rho_b - 1$, where ρ_b is the background density, can be inferred from the dynamical equations for the pressureless self-gravitating cosmic fluid.

In linear regime, i.e. for $\delta \ll 1$, the fluctuation field grows in time by an overall scale factor $D(t)$ (which depends on the cosmological parameter Ω), preserving its initial shape, $\delta(\mathbf{x}, t) = D(t)\,\delta(\mathbf{x}, t_i)$. As a result, the linear theory relation between the density and the velocity field is *local*

$$\delta(\mathbf{x}) = -f(\Omega)^{-1} H_0^{-1} \nabla \cdot \mathbf{v}(\mathbf{x})\,, \tag{1}$$

where H_0 is the Hubble constant and $f(\Omega) \simeq \Omega^{0.6}$ (see e.g. Peebles 1980). One can use the above formula to reconstruct from the large-scale velocity field the (linear) mass density field, up to an Ω-dependent multiplicative factor $f(\Omega)$. The comparison of the reconstructed mass field with the observed large-scale *galaxy* density field could therefore serve as a test for the gravitational instability hypothesis and as a method for estimating Ω (Dekel et al. 1993).

There are, however, both observational evidence and theoretical arguments for thinking that galaxies are biased tracers of mass. When the fluctuations are small one can assume that the galaxy and mass density contrast fields are linearly related, $\delta_g = b\,\delta$, hence

$$-\frac{1}{H_0} \nabla \cdot \mathbf{v}(\mathbf{x}) = \frac{f(\Omega)}{b} \delta_g(\mathbf{x})\,. \tag{2}$$

The comparison of the POTENT-reconstructed mass field with the IRAS galaxy field yields $f(\Omega)/b_{\rm IRAS}$ values close to unity (Dekel et al. 1993, Dekel 1994). It is then tempting to conclude that the large-scale dynamics is consistent with an assumption of $\Omega = 1$, provided that $b_{\rm IRAS}$ is also close to unity. However, since we do not know anything a priori about bias, we should measure it independently.

It has been suggested that nonlinear corrections to the linear density-velocity relation (hereafter DVR), equation (1), can help to perform such a measurement (Yahil 1991). The corrections are indeed necessary because there are points in the $\delta_{\rm POTENT}$-$\delta_{\rm IRAS}$ correlation diagram for which the density contrast reaches unity, clearly contradicting the underlying assumption of $\delta \ll 1$. On the other hand the rms fluctuation of the mass field, σ, is smaller (but not much smaller) than unity that means that the field is *weakly nonlinear*.

In weakly nonlinear regime perturbation theory can be efficiently applied, as the results of N-body simulations show (Juszkiewicz et al. 1995, Bernardeau 1994a,b, Baugh,

Gaztañaga & Efstathiou 1995, Łokas et al. 1995, Bernardeau & van de Weygaert 1996). However, most of the attempts to derive a weakly nonlinear extension of the linear DVR have been based on the Zel'dovich approximation and its modifications (Nusser et al. 1991, Gramman 1993). The Zel'dovich approximation is a useful qualitative guess of nonlinear dynamics but it provides only approximate answers for rigorously derived higher-order perturbative solutions. Consequently, it does much better than linear theory, but still does not predict accurately the weakly nonlinear relation between velocity and density, as verified by N-body simulations (Mancinelli et al. 1994, Ganon et al. 1996).

The first attempt, and so far the only one, to calculate DVR within the framework of rigorous Eulerian perturbation theory has been taken up by Bernardeau (1992a; hereafter B92). B92 has calculated the exact DVR for an unsmoothed final field in the limiting case of vanishing variance. The assumption $\sigma \to 0$ greatly simplifies mathematical calculations. It is not, however, well suited for the application to the IRAS-POTENT comparison: we are then not interested in the statistics of very rare events ($\delta \gg \sigma$) of linear field ($\sigma \ll 1$), but in the statistics of 'typical' events ($\delta \sim \sigma$) of a weakly nonlinear field ($\sigma \lesssim 1$). N-body cosmological simulations show that the exact formula of B92, when straightly applied to the case $\sigma \lesssim 1$, works worse than the Zel'dovich approximation (Mancinelli et al. 1994, Ganon et al. 1996).

B92 has also computed the first nonlinear (i.e. quadratic) correction for the DVR in the case of a smoothed final field with non-vanishing variance. However, neither the details of the derivation, nor the explicit form of the coefficient of the corrective term are given in the paper. On the other hand, a perturbation theory-inspired approximation of density as a third-order polynomial of velocity divergence turns out to be an excellent robust fit to N-body results (Mancinelli et al. 1994, Ganon et al. 1996). Theoretical construction of such a polynomial requires third order perturbative solutions and provides therefore higher order corrections to the DVR than those given by B92. All this inspired us to calculate the weakly nonlinear DVR of a smoothed final field with $\sigma \lesssim 1$ up to third order in perturbation theory.

The paper is organized as follows: in section 2 we derive weakly nonlinear DVR in its general form. In section 3 we compute values of the coefficients entering this relation for the case of scale-free power spectra, as well as for standard CDM. Discussion and concluding remarks are given in section 4.

2 General derivation of the density-velocity relation

In perturbation theory, one expands the solution for the density contrast as a series around the background value $\delta = 0$,

$$\delta = \delta_1 + \delta_2 + \delta_3 + \ldots\,, \tag{3}$$

and truncates it at some order. The linear theory solution mentioned in section 1 is just the perturbation theory series truncated at the lowest, i.e. first order term,

$$\delta_1(\mathbf{x},t) = D(t)\,\delta(\mathbf{x},t_i)\,. \tag{4}$$

Higher-order solutions are found iteratively: the second order contribution δ_2 is the solution to the dynamical equations with δ_1 as the source term for nonlinearities, and so on. Throughout this paper, we will consider only the growing modes, as the remaining ones are suppressed during linear evolution. In general, the n-th order solution is found to be of the order of $(\delta_1)^n$ (Fry 1984; Goroff et al. 1986). Let us define σ as the square root of the variance of the linear density field, i.e. $\sigma^2 = \langle\delta_1^2\rangle$, with $\langle\cdot\rangle$ meaning the ensemble averaging. We have $\delta_1 \sim \sigma$, $\delta_n \sim \sigma^n$, and the series (3) is a power series in a small parameter σ.

We describe the velocity field by a variable proportional to the velocity divergence,

$$\theta(\mathbf{x},t) \equiv -f(\Omega)^{-1} H_0^{-1} \nabla\cdot\mathbf{v}(\mathbf{x},t) \tag{5}$$

(which is slightly different from the commonly used definition, e.g. Bernardeau 1994a). The variable θ is as well expanded in a series

$$\theta = \theta_1 + \theta_2 + \theta_3 + \dots\,. \tag{6}$$

The linear theory solution, equation (1), therefore gives

$$\delta_1(\mathbf{x}) = \theta_1(\mathbf{x})\,. \tag{7}$$

Second order contributions to δ and θ are different. Their explicit dependence on Ω is extremely weak and in the range of cosmological interest, $0.1 \le \Omega \le 3$, the solutions are excellently approximated by the expressions which hold in the case of the Einstein-de Sitter universe (Bouchet et al. 1992), namely (Goroff et al. 1986)

$$\delta_2(\mathbf{x},t) = \frac{5}{7}\,\delta_1^2 + \partial_\alpha\delta_1\,\partial_\alpha\Delta_1 + \frac{2}{7}\partial_\alpha\partial_\beta\,\Delta_1\partial_\alpha\partial_\beta\Delta_1\,, \tag{8}$$

and

$$\theta_2(\mathbf{x},t) = \frac{3}{7}\,\delta_1^2 + \partial_\alpha\delta_1\,\partial_\alpha\Delta_1 + \frac{4}{7}\partial_\alpha\partial_\beta\,\Delta_1\partial_\alpha\partial_\beta\Delta_1\,. \tag{9}$$

Here, $\Delta_1(\mathbf{x},t)$ is the linear gravitational potential,

$$\Delta_1(\mathbf{x}) = -\int \frac{\mathrm{d}^3x'}{4\pi}\frac{\delta_1(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,. \tag{10}$$

We see therefore that up to second order in perturbation theory the divergence of the velocity field, $\nabla\cdot\mathbf{v}(\mathbf{x}) = -f(\Omega)H_0(\theta_1+\theta_2+\cdots)$, depends explicitly on Ω only via a multiplicative factor $f(\Omega)$.

Equation (7) means that if we read from a *linear* field the values of pairs $(\delta(\mathbf{x}),\,\theta(\mathbf{x}))$, point by point, and plot them on the δ-θ plane, they will lie on a straight line. The second

order contributions to $\delta(\mathbf{x})$ and $\theta(\mathbf{x})$, in addition to the local term $\sim \delta_1^2(\mathbf{x})$, contain nonlocal terms due to the linear gravitational potential, $\sim \sum \alpha_{\mathbf{x}'\mathbf{x}''}\delta_1(\mathbf{x}')\delta_1(\mathbf{x}'')$. As a result, the weakly nonlinear DVR is no longer local. However, given θ, the spread in the values of δ comes only from nonlinear corrections. Consequently, the points on the δ-θ plane are still strongly correlated: they are expected to form an elongated set of length $\sim \sigma$ and width $\sim \sigma^2$ around the mean trend (B92). This has been also observed in N-body simulations (Nusser et al. 1991, Bernardeau & van de Weygaert 1996). The mean trend can therefore serve as a very useful local approximation of a true nonlocal DVR.

Full information about the density-velocity correlation is contained in a joint probability distribution function (PDF) of weakly nonlinear variables δ and θ. Pioneering works on computing PDFs of weakly nonlinear cosmological density and velocity fields have been performed by Bernardeau. First, Bernardeau (1992b) computed a one-point PDF of an unsmoothed weakly nonlinear density field. Subsequently, he extended his calculations for the case of a top-hat window function (Bernardeau 1994a), and computed as well the PDF of a top-hat smoothed velocity divergence field (Bernardeau 1994b). A *joint* density-velocity PDF, however, is still known only for the case of an unsmoothed final field, and in the limit $\sigma \to 0$. B92 calculated it in the form of mean θ given δ; this relation is however easily invertible and the result is

$$\delta = \left(1 + \frac{2}{3}\theta\right)^{3/2} - 1\,. \tag{11}$$

Note that in the linear theory limit, $\theta \ll 1$, the above equation indeed reduces to equation (7).

Juszkiewicz et al. (1995) by means of N-body simulations have shown that a one-point PDF of a single variable δ (or θ) in the range of 'typical' events $\delta \sim \sigma$ can be very well approximated by the so-called Edgeworth series (e.g. Longuet-Higgins 1963, 1964 and references therein). The Edgeworth series is constructed from cumulants of the true distribution, defined as the connected part of the moments,

$$\kappa_n = \langle \delta^n \rangle_{conn}\,. \tag{12}$$

The cumulants of order $n > 2$ provide an effective measure of non-Gaussianity because for a Gaussian distribution they vanish. Throughout this paper we will assume Gaussian initial conditions. Consequently, all $n > 2$ cumulants of a fluctuation field are initially zero. During nonlinear phase of evolution, however, they acquire nonzero values. Fry (1984) showed that cumulants of cosmic density and velocity fields in weakly nonlinear regime obey the following scaling

$$\kappa_n = S_n \sigma^{2(n-1)} + \mathcal{O}(\sigma^{2n})\,. \tag{13}$$

To calculate the coefficient S_n, the perturbative solution of $(n-1)$th order is needed. Let

us define dimensionless quantities related to cumulants as follows

$$\lambda_n = \frac{\kappa_n}{\kappa_2^{n/2}} , \tag{14}$$

where by definition $\kappa_2 = \langle \delta^2 \rangle = \sigma_\delta^2$ is the nonlinear variance of a density field. From equation (13) one can deduce the order of weakly nonlinear corrections to its linear value

$$\sigma_\delta^2 = \sigma^2 + \mathcal{O}(\sigma^4) . \tag{15}$$

The coefficients λ_n are thus the cumulants of a standardized variable $\mu = \delta/\sigma_\delta$ and we will refer to them as to 'standard cumulants'. The first two nontrivial standard cumulants, λ_3 and λ_4, are called in statistics skewness and kurtosis, respectively.

The Edgeworth series reads

$$p(\mu) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\mu^2/2} \left[1 + \frac{1}{6}\lambda_3 H_3(\mu) + \frac{1}{24}\lambda_4 H_4(\mu) + \frac{1}{72}\lambda_3^2 H_6(\mu) + \cdots \right] \tag{16}$$

where $H_n(\mu)$'s are the n–th order Hermite polynomials generated by

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\mu^n} \mathrm{e}^{-\mu^2/2} = \mathrm{e}^{-\mu^2/2} H_n(\mu) . \tag{17}$$

In Table 1 we provide explicit forms of the few lowest order polynomials. From Equations (13), (14) and (15) we have

$$\lambda_n = S_n \sigma_\delta^{n-2} + \mathcal{O}(\sigma_\delta^n) , \tag{18}$$

which expresses the scaling behaviour of standard cumulants of a weakly nonlinear density field evolving from Gaussian initial conditions. In particular, $\lambda_3 = S_3 \sigma_\delta$ and $\lambda_4 = S_4 \sigma_\delta^2$, i.e. during weakly nonlinear evolution skewness and kurtosis grow like the rms fluctuation of the field and the square of it, respectively. In cosmology, there is a long tradition to call 'skewness' and 'kurtosis', respectively, the coefficients S_3 and S_4 themselves. We will honour it hereafter.

Equation (18) ensures that the Edgeworth series is a series expansion of an exact PDF in powers of a small parameter σ_δ (Longuet-Higgins 1963). In weakly nonlinear regime we can thus approximate the true PDF by the Edgeworth series truncated at some order. Using equation (18) the Edgeworth expansion, equation (16), can be rewritten in the explicitly perturbative, third-order form

$$p(\mu) = \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-\mu^2/2} \left[1 + \frac{1}{6} S_3 \sigma_\delta H_3(\mu) + \frac{1}{24} S_4 \sigma_\delta^2 H_4(\mu) + \frac{1}{72} S_3^2 \sigma_\delta^2 H_6(\mu) \right] . \tag{19}$$

The Edgeworth expansion for the variable θ or $\nu = \theta/\sigma_\theta$, where $\sigma_\theta^2 = \langle \theta^2 \rangle$ is the nonlinear variance of the velocity divergence field has the same form, except that S_3 and S_4 are then the skewness and the kurtosis of the velocity divergence field.

The third-order Edgeworth expansion describes accurately the shape of a true PDF up to $\mu \sim \sigma_\delta^{-1}$ (Juszkiewicz et al. 1995). The failure of the approximation in the very tails reflects the fact that it is constructed only from a few low order cumulants of the true distribution (see also Bernardeau & Kofman 1995). In the present paper, however, we are not interested in the statistics of very rare events in the δ-θ space. Instead, we want to calculate just the lowest conditional moment: mean δ given θ. For this purpose, we have to know the approximate form of the joint distribution for the variables δ and θ that needs to be accurate only for typical events, $\delta \sim \sigma_\delta$, $\theta \sim \sigma_\theta$. The two-point generalization of the above third-order Edgeworth series exactly satisfies this condition.

In fact, one can proceed in two ways. One can derive joint Edgeworth expansion and then calculate conditional moments from it. One can also, however, calculate the moments directly. Deriving the third-order joint Edgeworth expansion is a straightforward, but lengthy calculation, while of most interest for cosmology is just the first moment, describing the mean trend. Therefore, in this paper we calculate it directly, postponing the calculation of the joint Edgeworth expansion and higher-order moments (e.g. the variance around the mean trend) resulting from it to the next paper. The methods of calculating moments and the full PDF are still closely related and in the following calculation we are inspired to some extent by Longuet-Higgins (1963), who derived a *second-order* joint Edgeworth expansion in order to apply it to statistical theory of sea waves.

The conditional probability for δ given θ is

$$p(\delta)|_\theta = \frac{p(\delta,\theta)}{p(\theta)} \tag{20}$$

where $p(\delta,\theta)$ is the joint PDF for δ and θ. The characteristic function of $p(\delta,\theta)$ is

$$\Phi(it,is) = \int\!\!\int e^{it\delta+is\theta} p(\delta,\theta)\mathrm{d}\delta\mathrm{d}\theta\,. \tag{21}$$

Expanding the exponentials we obtain

$$\Phi(it,is) = \sum_{m,n=0}^{\infty} \frac{\langle \delta^m \theta^n \rangle}{m!n!} (it)^m (is)^n\,, \tag{22}$$

where $\langle \delta^m \theta^n \rangle$ are the joint moments of δ and θ,

$$\langle \delta^m \theta^n \rangle = \int\!\!\int \delta^m \theta^n p(\delta,\theta)\,\mathrm{d}\delta\mathrm{d}\theta\,. \tag{23}$$

If the joint moments are known, $p(\delta,\theta)$ can be calculated via the inverse Fourier transform,

$$p(\delta,\theta) = \frac{1}{(2\pi)^2}\int\!\!\int e^{-it\delta-is\theta}\Phi(it,is)\,\mathrm{d}t\mathrm{d}s\,. \tag{24}$$

Mean δ given θ, $\langle\delta\rangle|_\theta$, is by definition $\int \delta\, p(\delta)|_\theta \mathrm{d}\delta$. From equation (20) we have

$$\langle\delta\rangle|_\theta = \frac{\int \delta\, p(\delta,\theta)\,\mathrm{d}\delta}{p(\theta)}\,. \tag{25}$$

Let us denote $\int \delta\, p(\delta,\theta)\mathrm{d}\delta$ by $\mathcal{N}$. By equation (24),

$$\begin{aligned}\mathcal{N} &= \frac{1}{(2\pi)^2}\iiint e^{-it\delta-is\theta}\delta\,\Phi(it,is)\,\mathrm{d}t\mathrm{d}s\mathrm{d}\delta \\ &= -\frac{1}{(2\pi)^2}\iint e^{-is\theta}\Phi(it,is)\,\mathrm{d}t\mathrm{d}s\,\frac{\partial}{\partial(it)}\int e^{-it\delta}\mathrm{d}\delta \\ &= -\frac{1}{2\pi}\iint e^{-is\theta}\Phi(it,is)\,\frac{\partial}{\partial(it)}\delta_D(t)\,\mathrm{d}t\mathrm{d}s\,, \end{aligned} \tag{26}$$

where $\delta_D(t)$ denotes the Dirac delta function. Integrating by parts we obtain

$$\mathcal{N} = \frac{1}{2\pi}\int e^{-is\theta}\frac{\partial}{\partial(it)}\Phi(it,is)|_{t=0}\,\mathrm{d}s\,. \tag{27}$$

The characteristic function is related to the cumulant generating function, $\mathcal{K}$, by the equation

$$\Phi(it,is) = \exp\left[\mathcal{K}(it,is)\right]. \tag{28}$$

The cumulants, κ_{mn}, from which $\mathcal{K}$ is constructed,

$$\mathcal{K} = \sum_{(m,n)\neq(0,0)}^{\infty}\frac{\kappa_{mn}}{m!n!}(it)^m(is)^n\,, \tag{29}$$

are given by the *connected* part of the joint moments

$$\kappa_{mn} = \langle\delta^m\theta^n\rangle_{conn}\,. \tag{30}$$

Using equations (28) and (29) we obtain

$$\frac{\partial}{\partial(it)}\,\Phi(it,is)|_{t=0} = \left[\sum_{n=0}^{\infty}\frac{\kappa_{1n}}{n!}(is)^n\right]\,\exp\left[\sum_{n=1}^{\infty}\frac{\kappa_{0n}}{n!}(is)^n\right]. \tag{31}$$

By definition, κ_{0n} are the ordinary cumulants of the variable θ. The variables δ and θ have zero mean, so $\kappa_{10}=\kappa_{01}=0$. Equations (27) and (31) then give

$$\begin{aligned}\mathcal{N} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}s\,e^{-is\theta}\left[\sum_{n=1}^{\infty}\frac{\kappa_{1n}}{n!}(is)^n\right]\,\exp\left[\sum_{n=2}^{\infty}\frac{\kappa_{0n}}{n!}(is)^n\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}s\,e^{-\theta(is)+\frac{1}{2}\kappa_{02}(is)^2}\left[\sum_{n=1}^{\infty}\frac{\kappa_{1n}}{n!}(is)^n\right]\,\exp\left[\sum_{n=3}^{\infty}\frac{\kappa_{0n}}{n!}(is)^n\right]. \end{aligned} \tag{32}$$

Let us define a new variable $z=\kappa_{02}^{1/2}s$ and let us recall that μ and ν are the standardized variables,

$$\mu = \frac{\delta}{\sigma_\delta} = \frac{\delta}{\kappa_{20}^{1/2}} \qquad \text{and} \qquad \nu = \frac{\theta}{\sigma_\theta} = \frac{\theta}{\kappa_{02}^{1/2}}\,. \tag{33}$$

We then have

$$\mathcal{N} = \frac{1}{2\pi\kappa_{02}^{1/2}}\int_{-\infty}^{\infty}\mathrm{d}z\,e^{-\frac{1}{2}(z^2+2i\nu z)}\left[\sum_{n=1}^{\infty}\frac{\kappa_{1n}}{n!\kappa_{02}^{n/2}}(iz)^n\right]\,\exp\left[\sum_{n=3}^{\infty}\frac{\kappa_{0n}}{n!\kappa_{02}^{n/2}}(iz)^n\right]. \tag{34}$$

The standard joint cumulants are defined by

$$\lambda_{mn} = \frac{\kappa_{mn}}{\kappa_{20}^{m/2}\kappa_{02}^{n/2}}\,, \tag{35}$$

hence, $\kappa_{0n}/\kappa_{02}^{n/2} = \lambda_{0n}$, $\kappa_{1n}/\kappa_{02}^{n/2} = \kappa_{20}^{1/2}\lambda_{1n}$ and

$$\mathcal{N} = \frac{1}{2\pi}\left(\frac{\kappa_{20}}{\kappa_{02}}\right)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}z\, e^{-\frac{1}{2}(z^2+2i\nu z)}\left[\sum_{n=1}^{\infty}\frac{\lambda_{1n}}{n!}(iz)^n\right]\exp\left[\sum_{n=3}^{\infty}\frac{\lambda_{0n}}{n!}(iz)^n\right]. \tag{36}$$

One may ask why we have introduced the cumulant generating function: using just the characteristic function Φ, the above equation would look formally simpler. The reason is similar to that in case of constructing one-point PDF. From perturbation theory it follows that standard joint cumulants, equation (35), obey the following scaling hierarchy

$$\lambda_{mn} = S_{mn}\sigma^{m+n-2} + \mathcal{O}(\sigma^{m+n})\,. \tag{37}$$

where σ is the linear variance of δ or, equivalently, of θ (recall that at linear order $\delta = \theta$). The series in equation (36) are therefore power series in a small parameter σ and truncating them at some order p we neglect contributions which are $\sim \sigma^{p+1}$. Perturbation theory also predicts that

$$\sigma_\theta^2 = \langle\theta^2\rangle = \kappa_{02} = \sigma^2 + \mathcal{O}(\sigma^4)\,, \tag{38}$$

so when we are interested in the leading order terms in hierarchy (37) we can use linear σ instead of nonlinear σ_θ (or σ_δ, see eq. [15]).

In the present paper we want to calculate the weakly nonlinear extension of the linear δ-θ relation, up to cubic in θ, $\mathcal{O}(\sigma_\theta^3)$ terms. In equation (36), relating mean $\mu = \delta/\sigma_\delta$ and $\nu = \theta/\sigma_\theta$, we will thus keep terms up to the order of σ_θ^2. We have

$$\begin{aligned}\mathcal{N} &= \frac{1}{2\pi}\left(\frac{\kappa_{20}}{\kappa_{02}}\right)^{1/2}\int_{-\infty}^{\infty}\mathrm{d}z\, e^{-\frac{1}{2}(z^2+2i\nu z)} \\ &\times \left[\lambda_{11}(iz) + \frac{\lambda_{12}}{2}(iz)^2 + \frac{\lambda_{13}}{6}(iz)^3\right]\left[1 + \frac{\lambda_{03}}{6}(iz)^3 + \left\{\frac{\lambda_{04}}{24}(iz)^4 + \frac{\lambda_{03}^2}{72}(iz)^6\right\}\right]. \end{aligned} \tag{39}$$

In the expression above, $\lambda_{12} \sim \lambda_{03} \sim \sigma_\theta$ and $\lambda_{13} \sim \lambda_{04} \sim \lambda_{03}^2 \sim \sigma_\theta^2$. The cumulant λ_{11} deserves a separate, more detailed treatment. Defined as $\langle\delta\theta\rangle/[\langle\delta^2\rangle^{1/2}\langle\theta^2\rangle^{1/2}]$ (see eq.[35]), it is the correlation coefficient between the fields δ and θ. Since the fields are identical at first order, $\delta_1 = \theta_1$, at the lowest order $\lambda_{11} = 1$. From equation (37) it follows that the higher-order correction to this value of λ_{11} is $\mathcal{O}(\sigma_\theta^2)$, so in general $\lambda_{11} = 1 + \mathcal{O}(\sigma_\theta^2)$. Multiplying the polynomials in equation (39) we keep only the terms up to the order of σ_θ^2. It means also that we replace the products $\lambda_{11}\lambda_{mn}$ with $m + n \geq 3$ by λ_{mn}, since the correction is of at least cubic order in σ_θ. After sorting the resulting terms of the form $(iz)^n$ we integrate them, using the identity

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}z\, e^{-\frac{1}{2}(z^2+2i\nu z)}(iz)^n = H_n(\nu)\, e^{-\frac{1}{2}\nu^2} \tag{40}$$

where H_n are the n-th order Hermite polynomials. The result is

$$\begin{aligned}
\mathcal{N} &= \frac{1}{\sqrt{2\pi}}\left(\frac{\kappa_{20}}{\kappa_{02}}\right)^{1/2} e^{-\frac{1}{2}\nu^2} \\
&\times \left[\lambda_{11}H_1(\nu) + \frac{\lambda_{12}}{2}H_2(\nu) + \frac{\lambda_{03}}{6}H_4(\nu)\right. \\
&+ \left.\frac{\lambda_{13}}{6}H_3(\nu) + \left(\frac{\lambda_{04}}{24} + \frac{\lambda_{12}\lambda_{03}}{12}\right) H_5(\nu) + \frac{\lambda_{03}^2}{72}H_7(\nu)\right] . \qquad (41)
\end{aligned}$$

To calculate mean δ given θ from equation (25) we need also one-point PDF of velocity divergence. As already stated, it is given by the one-point Edgeworth series, equation (16), for the variable $\nu = \theta/\kappa_{02}^{1/2}$

$$p(\nu) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\nu^2/2}\left[1 + \frac{1}{6}\lambda_{03}H_3(\nu) + \frac{1}{24}\lambda_{04}H_4(\nu) + \frac{1}{72}\lambda_{03}^2H_6(\nu)\right] . \qquad (42)$$

Recalling that $\mu = \delta/\kappa_{20}^{1/2}$, and by equation (25), we have $\langle\mu\rangle|_\nu = \kappa_{20}^{-1/2}\langle\delta\rangle|_\theta = \kappa_{20}^{-1/2}\mathcal{N}/p(\theta)$ $= (\kappa_{02}/\kappa_{20})^{1/2}\mathcal{N}/p(\nu)$. From equations (41) and (42), expanding the denominator, multiplying, and keeping only the terms up to $\mathcal{O}(\sigma_\theta^2)$ we finally obtain

$$\langle\mu\rangle|_\nu = \lambda_{11}\nu + \mathcal{P}_2(\nu) + \mathcal{P}_3(\nu)\,, \qquad (43)$$

where

$$\mathcal{P}_2(\nu) = \frac{\lambda_{12}}{2}H_2(\nu) + \frac{\lambda_{03}}{6}\Big[H_4(\nu) - \nu H_3(\nu)\Big] \qquad (44)$$

and

$$\begin{aligned}
\mathcal{P}_3(\nu) &= \frac{\lambda_{13}}{6}H_3(\nu) + \frac{\lambda_{04}}{24}\Big[H_5(\nu) - \nu H_4(\nu)\Big] + \frac{\lambda_{12}\lambda_{03}}{12}\Big[H_5(\nu) - H_2(\nu)H_3(\nu)\Big] \\
&+ \frac{\lambda_{03}^2}{72}\Big[H_7(\nu) - \nu H_6(\nu) - 2H_3(\nu)H_4(\nu) + 2\nu H_3^2(\nu)\Big] . \qquad (45)
\end{aligned}$$

Note that $\mathcal{P}_2(\nu)$ is $\mathcal{O}(\sigma_\theta)$, and $\mathcal{P}_3(\nu)$ is $\mathcal{O}(\sigma_\theta^2)$. (Remember that ν itself is standardized, so $\nu \sim \mathcal{O}(\sigma_\theta^0) = \mathcal{O}(1)$ and σ_θ-dependence comes only from the standard cumulants.)

Equation (43) expresses weakly nonlinear density-velocity relation (DVR). In linear regime, $\sigma_\theta \to 0$, the correlation coefficient between δ and θ is unity, $\lambda_{11} = 1$. Moreover, in this limit $\mathcal{P}_2$ and $\mathcal{P}_3$ also approach zero, so from equation (43) we reobtain the linear theory result, $\langle\mu\rangle|_\nu = \nu$, or $\langle\delta\rangle|_\theta = \theta$. The polynomials $\mathcal{P}_2$ and $\mathcal{P}_3$ are higher-order corrections to this linear relation. As we took into account the corrections up to third order in perturbation theory, we do not expect terms of order higher than cubic in ν to appear in equations (44)-(45). Indeed, in these equations all the terms of the form ν^n with $n > 3$ remarkably cancel out. To prove this, we use the recurrent relation for the Hermite polynomials

$$H_n(\nu) - \nu H_{n-1}(\nu) = -(n-1)H_{n-2}(\nu)\,, \qquad (46)$$

along with their explicit forms for $n = 1, \dots, 5$ (see Table 1). The result is

$$\mathcal{P}_2(\nu) = \frac{\lambda_{12} - \lambda_{03}}{2}(\nu^2 - 1)\,, \tag{47}$$

and

$$\mathcal{P}_3(\nu) = \frac{\lambda_{13} - \lambda_{04}}{6}(\nu^3 - 3\nu) + \frac{(\lambda_{03} - \lambda_{12})\lambda_{03}}{2}(\nu^3 - 2\nu)\,. \tag{48}$$

We see that $\mathcal{P}_2(\nu)$ and $\mathcal{P}_3(\nu)$ contain quadratic and cubic corrections in ν, respectively.

Let us now deal with the joint cumulants in $\mathcal{P}_2$. We have $\lambda_{03}\sigma_\theta^3 = \kappa_{03} = \langle\theta^3\rangle = S_{3\theta}\sigma_\theta^4$, where $S_{3\theta}$ denotes the skewness of the variable θ. Thus we get

$$\lambda_{03} = S_{3\theta}\sigma_\theta\,. \tag{49}$$

The other, mixed cumulant in $\mathcal{P}_2(\nu)$ is defined as $\lambda_{12}\sigma_\theta^3 = \langle(\delta_1 + \delta_2 + \cdots)(\theta_1 + \theta_2 + \cdots)^2\rangle$. Recalling that $\delta_1 = \theta_1$, we obtain

$$\lambda_{12} = \left[\frac{1}{3}S_{3\delta} + \frac{2}{3}S_{3\theta}\right]\sigma_\theta\,, \tag{50}$$

where $S_{3\delta}$ denotes the skewness of the variable δ. Thus λ_{12} is a linear combination of ordinary third-order cumulants of the single variables δ and θ. Equations (47), (49) and (50) then yield

$$\mathcal{P}_2(\nu) = \frac{\Delta S_3}{6}\sigma_\theta(\nu^2 - 1)\,, \tag{51}$$

where

$$\Delta S_3 = S_{3\delta} - S_{3\theta}\,. \tag{52}$$

We can now calculate the lowest order weakly nonlinear extension of the linear DVR. As we will show later, $\lambda_{11} = 1 + \mathcal{O}(\sigma_\theta^2)$. Keeping the terms up to $\mathcal{O}(\sigma_\theta)$ in equation (43) we thus have

$$\langle\mu\rangle|_\nu = \left[1 + \mathcal{O}(\sigma_\theta^2)\right]\nu + \frac{\Delta S_3}{6}\sigma_\theta(\nu^2 - 1) + \mathcal{O}(\sigma_\theta^2)\,, \tag{53}$$

or

$$\langle\mu\rangle|_\nu = \nu + \frac{\Delta S_3}{6}\sigma_\theta(\nu^2 - 1) + \mathcal{O}(\sigma_\theta^2)\,. \tag{54}$$

¿From equation (38) it follows that $\sigma_\delta = \sigma_\theta + \mathcal{O}(\sigma_\theta^3)$, so $\mu = \delta/\sigma_\delta = \delta/\sigma_\theta + \mathcal{O}(\sigma_\theta^2)$, hence

$$\langle\delta\rangle|_\nu = \sigma_\theta\nu + \frac{\Delta S_3}{6}\sigma_\theta^2(\nu^2 - 1) + \mathcal{O}(\sigma_\theta^3)\,. \tag{55}$$

Since $\nu = \theta/\sigma_\theta$ we end up with

$$\langle\delta\rangle|_\theta = \theta + \frac{\Delta S_3}{6}(\theta^2 - \sigma_\theta^2) + \mathcal{O}(\sigma_\theta^3)\,. \tag{56}$$

The above equation is the lowest, second order weakly nonlinear extension of the linear DVR. Consequently, the coefficient of the corrective term is composed from cumulants calculable at second order ($S_{3\delta}$ and $S_{3\theta}$), and the term is quadratic in θ. This term is shifted

down additionally by σ_θ^2. Note from equation (25) that $\int \langle\delta\rangle|_\theta p(\theta)\mathrm{d}\theta = \int \delta\, p(\delta,\theta)\,\mathrm{d}\delta\mathrm{d}\theta = \int \delta\, p(\delta)\,\mathrm{d}\delta = \langle\delta\rangle$ (not conditional, but ordinary), that is zero by definition. The σ_θ^2 term, naturally emerging from our calculations, precisely ensures this.

Apart from deriving the exact DVR in the case of an unsmoothed field with vanishing variance, equation (11), B92 calculated also the second-order DVR including the effects of finite variance and smoothing of a final field. Our result, equation (56), coincides exactly with equation (17) of B92 with the coefficient $B = \Delta S_3/6$.

We will deal now with the cumulants in the $\mathcal{P}_3$ term. From equation (49)-(50) we have

$$\frac{(\lambda_{03}-\lambda_{12})\lambda_{03}}{2} = -\frac{\Delta S_3\, S_{3\theta}}{6}\sigma_\theta^2\,. \tag{57}$$

The joint cumulant λ_{13}, unlike λ_{12}, is not a linear combination of ordinary cumulants of the single variables δ and θ. We have $(\lambda_{13}-\lambda_{04})\sigma_\theta^4 = \langle\delta\theta^3\rangle - \langle\theta^4\rangle = \langle(\delta-\theta)\theta^3\rangle = \langle(\delta_2-\theta_2\, + \delta_3-\theta_3+\cdots)(\theta_1+\theta_2+\theta_3+\cdots)^3\rangle$ and therefore

$$\lambda_{13}-\lambda_{04} = \Sigma_4\sigma_\theta^2\,, \tag{58}$$

where

$$\Sigma_4 = \frac{3\langle\theta_1^2\theta_2(\delta_2-\theta_2)\rangle + \langle\delta_1^3\delta_3\rangle - \langle\theta_1^3\theta_3\rangle}{\sigma^6}\,. \tag{59}$$

In the expression above $\langle\cdot\rangle$ stands for the *connected* part of the moments. Note that Σ_4 is *not* equal to $S_{4\delta}-S_{4\theta}$: while the last two terms in equation (59) are indeed parts of the expressions for the ordinary kurtosis of a single variable δ or θ, respectively, the first term is a truly mixed moment and constitutes a new quantity. Using the results (57)-(58) in equation (48) we obtain

$$\mathcal{P}_3(\nu) = \frac{\Sigma_4 - \Delta S_3\, S_{3\theta}}{6}\,\sigma_\theta^2\nu^3 + \left[\frac{\Delta S_3\, S_{3\theta}}{3} - \frac{\Sigma_4}{2}\right]\sigma_\theta^2\nu\,. \tag{60}$$

Equation (43) expresses weakly nonlinear extension of the linear DVR up to $\mathcal{O}(\sigma_\theta^2)$ corrections. The scaling of the standard cumulants with σ_θ, equation (37), ensures that it was enough to calculate λ_{mn} with $m+n\geq 3$ at the lowest order. The corrections to λ_{11}, however, are $\mathcal{O}(\sigma_\theta^2)$, so they cannot be neglected. Similarly, $\mu = \delta/\kappa_{20}^{1/2} = \delta/\sigma_\delta = (\sigma_\theta/\sigma_\delta)(\delta/\sigma_\theta) = [1+\mathcal{O}(\sigma_\theta^2)]\delta/\sigma_\theta$, so the corrections to the linear evolution of the variance of δ and θ should be taken into account as well. Changing the variables in equation (43) to δ and θ, equation (33), we have

$$\langle\delta\rangle|_\theta = \left(\frac{\kappa_{20}}{\kappa_{02}}\right)^{1/2}\lambda_{11}\theta + \sigma_\theta[\mathcal{P}_2(\theta/\sigma_\theta)+\mathcal{P}_3(\theta/\sigma_\theta)] + \mathcal{O}(\sigma_\theta^4)\,. \tag{61}$$

By definitions (35) and (30),

$$\left(\frac{\kappa_{20}}{\kappa_{02}}\right)^{1/2}\lambda_{11} = \frac{\kappa_{11}}{\kappa_{02}} = \frac{\langle\delta\theta\rangle}{\langle\theta^2\rangle} = \frac{\langle\delta\theta\rangle - \langle\theta^2\rangle + \langle\theta^2\rangle}{\langle\theta^2\rangle} = 1 + \frac{\langle(\delta-\theta)\theta\rangle}{\langle\theta^2\rangle}\,, \tag{62}$$

which after expanding δ and θ in perturbative series gives

$$\left(\frac{\kappa_{20}}{\kappa_{02}}\right)^{1/2} \lambda_{11}\theta = [1 + \Sigma_2 \sigma_\theta^2]\,\theta\,, \tag{63}$$

where

$$\Sigma_2 = \frac{\langle \theta_2(\delta_2 - \theta_2)\rangle + \langle \delta_1\delta_3\rangle - \langle \theta_1\theta_3\rangle}{\sigma^4}\,. \tag{64}$$

Again, Σ_2 is *not* equal to $\langle \delta^2\rangle - \langle \theta^2\rangle$. The last two terms in equation (64) are indeed parts of the expressions for nonlinear corrections to the linear evolution of variance of δ and θ, but the first term is a truly mixed moment and constitutes a new quantity.

To obtain third-order weakly nonlinear DVR in its final form we combine equations (51), (60) and (63) with (61). Note that $\mathcal{P}_3$ contains also a term linear in θ. We have

$$\langle \delta\rangle|_\theta = a_1\theta + a_2(\theta^2 - \sigma_\theta^2) + a_3\theta^3\,, \tag{65}$$

where

$$a_1 = 1 + \left[\Sigma_2 + \frac{\Delta S_3\, S_{3\theta}}{3} - \frac{\Sigma_4}{2}\right]\sigma_\theta^2\,, \tag{66}$$

$$a_2 = \frac{\Delta S_3}{6}\,, \tag{67}$$

$$a_3 = \frac{\Sigma_4 - \Delta S_3\, S_{3\theta}}{6}\,, \tag{68}$$

with Σ_2 and Σ_4 given by equations (64) and (59), respectively. Equations (65)-(68) constitute the main result of this section. Note that we reobtain the second-order DVR, equation (56), when we neglect the $\mathcal{O}(\sigma_\theta^3)$ terms (i.e. $\sim \sigma_\theta^2\theta$ and $\sim \theta^3$).

An important conclusion can be drawn immediately: the DVR of a weakly nonlinear ($\sigma_\theta \lesssim 1$) field is different from the linear theory prediction, equation (7), even for $|\theta| \ll 1$. Namely, in this case

$$\langle \delta\rangle|_\theta = a_1\theta - a_2\sigma_\theta^2\,. \tag{69}$$

Thus, the linear relation is shifted down, as already discussed. What is perhaps even more interesting, the coefficient a_1 generally departs from unity. The strength of this shift and departure depends however on the particular values of the coefficients a_n. This is the subject of the next section.

3 Numerical calculations

A brief outline of the perturbative solutions to the Newtonian equations of motion needed for numerical calculations of the coefficients a_n is given in Appendix A.

The smoothing of the fields on scale R is introduced by the convolution of the density contrast (or velocity divergence) field and the filtering function W

$$\delta_R(\mathbf{x}, t) = \int \mathrm{d}^3y\, \delta(\mathbf{y}, t)\, W(|\mathbf{x} - \mathbf{y}|, R)\,. \tag{70}$$

We perform our calculations for a Gaussian filter function which is appropriate for observational analysis of cosmic velocity fields and comparing them with the density fields (e.g. POTENT and IRAS-POTENT comparison). The Fourier representation of the Gaussian window function is given by

$$W(kR) = \mathrm{e}^{-k^2R^2/2}. \tag{71}$$

We assume a Gaussian distribution for the first order δ_1 and θ_1 and define

$$\sigma^2 = \langle \delta_1^2 \rangle = D^2(t) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} P(k)\, W^2(kR) \tag{72}$$

as the linear variance of the density (velocity divergence) field. We assume that for $\sigma < 1$, the first few terms in the perturbative expansion provide a good approximation of the exact solution. Since δ_1 and θ_1 are assumed to be Gaussian random fields, all their statistical properties as well as those of the higher order terms in the perturbative series are determined by the power spectrum $P(k)$, defined as

$$\langle \delta_1(\mathbf{p}) \delta_1(\mathbf{q}) \rangle \;=\; (2\pi)^3 \delta_D\,(\mathbf{p}+\mathbf{q})\, P(p). \tag{73}$$

3.1 The calculation of the coefficient a_2

The values of skewness for density contrast and velocity divergence fields, given to the lowest perturbative order respectively by

$$S_{3\delta} = \frac{3\langle \delta_1^2 \delta_2 \rangle}{\sigma^4} \tag{74}$$

and

$$S_{3\theta} = \frac{3\langle \theta_1^2 \theta_2 \rangle}{\sigma^4}\,, \tag{75}$$

depend on the assumed form of the power spectrum.

We begin by considering spectra with a power-law form

$$P(k) = C k^n, \quad -3 \le n \le 1, \tag{76}$$

where C is a normalization constant. For such fields, smoothed with a Gaussian filter, the linear order contribution to the variance given by equation (72) is

$$\sigma^2 = C D^2(t) \frac{\Gamma(\frac{n+3}{2})}{(2\pi)^2 R^{n+3}}, \tag{77}$$

where R is the smoothing scale. The values of skewness are (Łokas et al. 1995)

$$S_{3\delta} = 3\; {}_2F_1\left(\frac{n+3}{2}, \frac{n+3}{2}, \frac{3}{2}, \frac{1}{4}\right) - \left(n + \frac{8}{7}\right)\; {}_2F_1\left(\frac{n+3}{2}, \frac{n+3}{2}, \frac{5}{2}, \frac{1}{4}\right) \tag{78}$$

$$S_{3\theta} = 3\; {}_2F_1\left(\frac{n+3}{2}, \frac{n+3}{2}, \frac{3}{2}, \frac{1}{4}\right) - \left(n + \frac{16}{7}\right)\; {}_2F_1\left(\frac{n+3}{2}, \frac{n+3}{2}, \frac{5}{2}, \frac{1}{4}\right) \tag{79}$$

where ${}_2F_1$ is the hypergeometric function. Therefore the coefficient a_2 is

$$a_2 = \frac{S_{3\delta} - S_{3\theta}}{6} = \frac{4}{21}\ {}_2F_1\left(\frac{n+3}{2}, \frac{n+3}{2}, \frac{5}{2}, \frac{1}{4}\right). \tag{80}$$

The result is very weakly dependent on the value of the Ω parameter (for details see the appendix in Łokas et al. 1995; Bernardeau et al. 1995 and Bouchet et al. 1992). A good approximation for the Ω dependence in the range $0.1 \leq \Omega \leq 3$ is obtained by replacing the constant coefficient 4/21 in equation (80) with an Ω-dependent function

$$G(\Omega) = \frac{1 - 2C(\Omega) + K(\Omega)}{3} \tag{81}$$

where

$$K(\Omega) = \frac{3}{7}\Omega^{-2/63}, \quad C(\Omega) = \frac{3}{7}\Omega^{-1/21}. \tag{82}$$

The second column of Table 2 gives the values of a_2 for integer and half-integer values of the spectral index n and $\Omega = 1$ while Figure 1 shows the coefficient a_2 as a function of n for three different values of Ω.

We have chosen the scale-free spectra of the form (76) not only because of their simplicity but also for their straightforward applicability to realistic power spectra. Indeed, in the case of higher order cumulants the value of the cumulant (the skewness or the kurtosis) is very well approximated by the result for the scale-free spectra with the effective index defined as (Bernardeau 1994a)

$$n_{eff} = -\frac{R}{\sigma^2}\frac{d\sigma^2(R)}{dR} - 3. \tag{83}$$

As an example of a scale-dependent power spectrum we consider the standard CDM spectrum

$$P(k) = \frac{Ck^n}{\left\{1 + \left[l_1 k/\Gamma + (l_2 k/\Gamma)^{3/2} + (l_3 k/\Gamma)^2\right]^\nu\right\}^{2/\nu}} \tag{84}$$

with $n = 1$, $\Gamma = 0.5$, $\nu = 1.13$ and the constants in units of h^{-1} Mpc are $l_1 = 6.4$, $l_2 = 3.0$, $l_3 = 1.7$ (e.g. Efstathiou, Bond & White, 1992). We normalize the spectrum so that the linear rms density fluctuation in spheres of radius $R = 8h^{-1}$ Mpc is equal to unity. Thus the definition of variance (72) together with the following shape of the spherical top hat window function in Fourier space

$$W_{TH}(kR) = 3\sqrt{\frac{\pi}{2}}(kR)^{-3/2}J_{3/2}(kR) \tag{85}$$

(where J is the Bessel function) and the power spectrum (84) yield the normalization constant of $C = 4.09 \times 10^5$ $(h^{-1}$ Mpc$)^4$. Note, that the normalization procedure is the only place where we use the top hat filter, all other calculations are performed for a Gaussian window function (71).

We have calculated the coefficient a_2 for the CDM spectrum at different smoothing scales R in two ways. First we have found numerically the exact values of $S_{3\delta}$ and $S_{3\theta}$ for CDM spectrum following the procedure of Łokas et al. (1995). These values are shown in the third and fourth column of Table 3. They produce the exact value of the coefficient a_2 given in the fifth column of the Table. The second option was to use the formula (80) with n_{eff} corresponding to each scale, calculated from equation (83). The second column of Table 3 gives the values of the effective index corresponding to each of the scales. Thus obtained value of a_2 is given in the last column of Table 3. The comparison between the two values of a_2 shows clearly that the discrepancy between them is less than 1% at all scales. The exact values of a_2 for the CDM are repeated in Table 4, which summarizes the results for this spectrum.

3.2 The calculation of the coefficient a_3

As we have shown in the previous section, the coefficient a_3 is given by

$$a_3 = \frac{\Sigma_4 - \Delta S_3 S_{3\theta}}{6} \tag{86}$$

where

$$\Sigma_4 = \frac{3\langle\delta_1^2\delta_2\theta_2\rangle - 3\langle\theta_1^2\theta_2^2\rangle + \langle\delta_1^3\delta_3\rangle - \langle\theta_1^3\theta_3\rangle}{\sigma^6}. \tag{87}$$

We have named the quantity Σ_4 to stress its similarity to the kurtosis of density and velocity divergence fields, which to lowest order in perturbation theory are given respectively by

$$S_{4\delta} = \frac{6\langle\delta_1^2\delta_2^2\rangle + 4\langle\delta_1^3\delta_3\rangle}{\sigma^6} \tag{88}$$

and

$$S_{4\theta} = \frac{6\langle\theta_1^2\theta_2^2\rangle + 4\langle\theta_1^3\theta_3\rangle}{\sigma^6} \tag{89}$$

(unless the initial conditions are non-Gaussian: for details in the latter case see Chodorowski & Bouchet 1996). This shows that most of the expressions constituting the value of Σ_4 for power law spectra has already been calculated by Łokas et al. (1995) while performing the calculations for kurtosis. Since they were not published, we give them in Table 5 for integer and half integer values of the spectral index n. They will also be needed in the calculations of the next subsection. The only unknown part of Σ_4 is the expression of the form $\langle\delta_1^2\delta_2\theta_2\rangle$ which is calculated in Appendix B.

In the case of no smoothing (when window function $W(kR) = 1$), which also corresponds to putting the spectral index $n = -3$, a completely analytic result can be obtained fairly easily; we get $\langle\delta_1^2\delta_2\theta_2\rangle/\sigma^6 = 1768/441 \approx 4.01$. In this case we have

$$a_3 = -\frac{40}{3969} \approx -0.0101 \tag{90}$$

The sixth column of Table 5 gives the values of Σ_4 calculated according to (87). Finally, the third column of Table 2 lists the values of a_3 obtained from equation (86) in the case of power-law spectra for integer and half integer values of the spectral index n.

Bernardeau (1994a) showed that third order solutions for δ and θ, similarly to second order, depend very weakly on Ω. Since the coefficient a_3 is constructed from terms up to third order, its expected dependence on Ω is very weak. Bernardeau (1994a) did not give explicit forms for weakly Ω-dependent third order solutions. Unlike to the case of the coefficient a_2 (see previous subsection), we cannot therefore verify in detail the above supposition. Still, we are able to do this at least for the case of the spectral index $n = -3$. The exact formula of B92, equation (11), describes the limiting case $\sigma^2 \to 0$, $n = -3$ and $\Omega = 0$. One cannot thus use it to deduce the value of the coefficient a_1 which includes corrections $\mathcal{O}(\sigma^2)$. On the other hand, one can use this formula to derive the values of the coefficients a_2 and a_3, which are calculated in the limit $\sigma^2 \to 0$. The Taylor expansion of equation (11) yields

$$a_2(\Omega = 0) = \frac{1}{6} \approx 0.167 \tag{91}$$

and

$$a_3(\Omega = 0) = -\frac{1}{54} \approx -0.0185\,. \tag{92}$$

The corresponding values for a_2 and a_3 in the case $n = -3$, $\Omega = 1$ are respectively 0.190 and -0.0101 (Table 2). The relative change of the value of the coefficient a_3 is therefore greater than the relative change of a_2. Nevertheless, also a_3 depends on Ω extremely weakly in a sense that it almost vanishes *both* for $\Omega = 1$ and $\Omega = 0$. All kurtosis-type quantities entering the definition of a_3 are of order of unity (see Table 5) and very precise cancellation of them is needed to assure $a_3 \ll 1$. Therefore even weak dependence of the perturbative solutions on Ω could in principle destroy this 'fine tuning'. For $n = -3$ this is clearly not the case.

In fact we were able to check it rigorously for all values of n for the kurtosis-type quantities in Σ_4 (eq. [87]) that involve only second order solutions. In the range of $0.1 < \Omega < 3$ we found that the Ω-dependence of $\langle \delta_1^2 \delta_2 \theta_2 \rangle$ and $\langle \theta_1^2 \theta_2^2 \rangle$ is similar and almost cancels out when the two are subtracted. Analogously, Bernardeau (1994a) noted that the combination $S_{4\theta}/S_{3\theta}^2$ is almost independent on Ω, to much bigger extent than the moments $S_{3\theta}$ and $S_{4\theta}$ themselves. Very weak dependence of a_3 on Ω is an interesting problem and we will address it in more detail elsewhere.

To obtain the values of the coefficient a_3 for the CDM spectrum we apply the effective index method described in the previous subsection. For each of the indices calculated from equation (83) for a given Gaussian smoothing scale we interpolate the value of a_3 from the values given in Table 2 using an accurate polynomial fit. The results are presented in the last column of Table 4.

3.3 The calculation of the coefficient a_1

It has been proved that the coefficient a_1 is given by

$$a_1 = 1 + \left[\Sigma_2 + \frac{\Delta S_3 S_{3\theta}}{3} - \frac{\Sigma_4}{2}\right] \sigma_\theta^2, \tag{93}$$

where

$$\Sigma_2 = \frac{\langle\delta_2\theta_2\rangle - \langle\theta_2^2\rangle + \langle\delta_1\delta_3\rangle - \langle\theta_1\theta_3\rangle}{\sigma^4} \tag{94}$$

and Σ_4 has been defined and discussed in the previous subsection.

The quantities involved in Σ_2 are of the form of the lowest order weakly nonlinear corrections to the variance of the density and velocity divergence fields which are of the order of σ^4 (Łokas et al. 1996)

$$\frac{\sigma_\delta^2 - \sigma^2}{\sigma^4} = \frac{\langle\delta_2^2\rangle + 2\langle\delta_1\delta_3\rangle}{\sigma^4} \tag{95}$$

$$\frac{\sigma_\theta^2 - \sigma^2}{\sigma^4} = \frac{\langle\theta_2^2\rangle + 2\langle\theta_1\theta_3\rangle}{\sigma^4}. \tag{96}$$

We recall that σ_δ^2 and σ_θ^2 stand for nonlinear variance of density and velocity divergence respectively while σ^2 is the linear variance given by equation (72). In equation (93) σ_θ^2 can be replaced by σ^2 since their difference is already $\mathcal{O}(\sigma_\theta^4)$.

The details of calculations of the terms involved in Σ_2 are given in Appendix C. As discussed in the Appendix, some of the terms are divergent at spectral indices $n > -1$. Instead of dwelling on those divergences we focused our attention on analysis concerning scale-free power spectra in the case of no smoothing and the case of $-2 \leq n < -1$ with smoothing, which is well justified observationally. As recent analyses of measurements suggest (Gaztañaga 1994; Feldman, Kaiser & Peacock 1994; Peacock & Dodds 1994) the linear power spectrum can be approximated over large range of scales by a power law of spectral index $n = -1.4 \pm 0.1$.

Calculated at the lowest order, the values of cumulants of an unsmoothed field do not depend on the underlying power spectrum and are equal to the values calculated for a smoothed field with spectral index $n = -3$. This is not, however, the case for higher-order corrections to their values. In the case of scale-free power spectra (76) when no smoothing is applied ($W(kR) = 1$) we obtain (see Appendix C)

$$\Sigma_2 = \frac{1297}{4410} + h(n) \approx 0.3 \tag{97}$$

where $h(n)$ is the part weakly dependent on the spectral index n which increases the rational number by roughly 10%. The remaining terms in the expression for the coefficient a_1, equation (93), are skewness and kurtosis-related quantities and it was sufficient to calculate them at the lowest order. Combining equation (97) with the values of these terms

corresponding to the no smoothing case (i.e. the values for $n = -3$ in Table 2 and Table 5 and the skewness values from equation (79)) we finally get

$$a_1 \approx 1 - 0.4\,\sigma^2. \tag{98}$$

When smoothing is introduced, for $n = -2$ we obtain (see Appendix C)

$$\Sigma_2 = \frac{23}{196}\pi \approx 0.369. \tag{99}$$

Combined with the other numbers calculated for $n = -2$ this result yields

$$a_1 = 1 - 0.172\,\sigma^2. \tag{100}$$

In the range $-2 \leq n < -1$ we calculate Σ_2 numerically and Σ_4 by interpolating the values given in Table 5. The calculation provides an independent check of the result for $n = -2$, equation (99), obtained analytically. Table 6 shows the two corrections to a_1 separately: while the part containing Σ_4 remains roughly constant, Σ_2 grows with n until it blows up at $n = -1$. The last column of Table 6 lists the values of a_1 in the σ-dependent way in order not to obscure the results by choosing arbitrary normalization needed for estimating σ. Since the value of σ^2 is of the order of unity on the scales of interest, it is clear from Table 5 that at the observationally preferred spectral index $n \approx -1.4$ the value of a_1 significantly departs from unity. It must be noted, however, that the nonlinear correction strongly depends on n and reaches zero between $n = -1.6$ and $n = -1.7$.

To provide an example of the values of a_1 we have normalized the power law spectra so that linear rms fluctuation in spheres of radius $8\,h^{-1}$ Mpc is equal to unity. The resulting values of σ^2 and a_1 for spectral indices $n = -1.4 \pm 0.1$ at two different Gaussian smoothing scales $R = 5\ h^{-1}$ Mpc and $R = 12\ h^{-1}$ Mpc are listed in Table 7.

Due to the reasons mentioned in the previous subsection we cannot explicitly examine the dependence of the coefficient a_1 on Ω. Still, it is constructed from moments involving second and third order solutions which have been proved to depend on Ω very weakly. Consequently, the expected dependence of a_1 on Ω is weak.

As an example of a scale-dependent power spectrum we again adopt the standard CDM model which, because of its behaviour at large wave-numbers ($P(k) \propto k^{-3}$), does not introduce any challenges in the integration. The values of Σ_2 can be calculated numerically for a given smoothing scale. By combining with skewness values from Table 3 and the interpolated kurtosis-type values from Table 5 we end up with the coefficient a_1 for the CDM spectrum. The values for different smoothing scales are given in the last column of Table 4. Although the Σ_2 values grow with scale (the remaining input to the correction to a_1 remains roughly constant for this range of scales, see the last column of Table 5), the σ^2 values decrease much faster and, as we would expect for the perturbative results, at larger (i.e. more linear) scales the coefficient a_1 approaches its linear value, unity.

Tables 2, 4, 6 and 7 and Figures 1, 2 and 3 summarize the main results of this section. Table 2 provides the values of the coefficients a_2 and a_3 for power law spectra in the whole range of the spectral index: $-3 \le n \le 1$. Those results are plotted in Figure 1 which also shows the Ω-dependence of the coefficient a_2. The values of the coefficient a_1 for power law spectra and the range of spectral index $-2 \le n < -1$ are given in Table 6. The correction to unity divided by σ^2 is plotted in Figure 2. Table 7 lists the numerical values of a_1 at $n = -1.4 \pm 0.1$ for two different smoothing scales when the $\sigma_8 = 1$ (top hat) normalization is adopted. The coefficients a_1, a_2 and a_3 for the standard CDM spectrum for a wide range of smoothing scales are provided in Table 4. Figure 3 shows their dependence on smoothing radius in the weakly nonlinear range of scales.

4 Disentangling Ω and linear bias

The bottom line of our calculations is to propose a method, based on nonlinear corrections to the linear DVR, for measuring independently Ω and bias from an IRAS-POTENT-like, density-velocity comparison. Let us assume that the galaxy and mass density contrast fields are linearly related, i.e. $\delta_g = b\,\delta$, or

$$\delta = b^{-1}\delta_g \,. \tag{101}$$

We introduce a new variable

$$\delta_v = -\frac{1}{H_0}\nabla\cdot\mathbf{v}\,. \tag{102}$$

By definition (5) we have

$$\theta = f^{-1}(\Omega)\,\delta_v\,. \tag{103}$$

For the sake of simplicity let us consider the case in which a cosmic field is smoothed over a sufficiently large volume that the third order corrections to the weakly nonlinear DVR, equation (65), can be neglected. Using equations (101) and (103) we can rewrite DVR in the form relating two observables: the galaxy density contrast, δ_g, and the (minus) divergence of the velocity field, δ_v. We have

$$\langle\delta_g\rangle_{|\delta_v} = \frac{b}{f}\delta_v + a_2\frac{b}{f^2}(\delta_v^2 - \sigma_v^2)\,. \tag{104}$$

In the previous section we showed that the coefficient a_2 practically does not depend on Ω. One can thus propose the following method for disentangling the effects of Ω and linear bias. First, as the output of POTENT take simply δ_v (i.e. without any corrections for nonlinearity). Next, plot the diagram δ_g–δ_v. Finally, fit to the points a second order polynomial,

$$\delta_g = c_1\delta_v + c_2(\delta_v^2 - \sigma_v^2)\,. \tag{105}$$

Comparing equation (105) with (104) we see that the fitted coefficients c_1 and c_2 are related to f and b by

$$c_1 = \frac{b}{f}\,, \tag{106}$$

and

$$c_2 = a_2 \frac{b}{f^2}\,. \tag{107}$$

So far, only the linear coefficient c_1 has been measured. The results are usually expressed in terms of the variable (Dekel et al. 1993)

$$\beta = c_1^{-1}\,. \tag{108}$$

The difficulty that from the linear density-velocity comparison one can estimate only the ratio $f(\Omega)/b$, or $b/f(\Omega)$, is sometimes called the 'Ω-bias degeneracy problem'. In our opinion, however, there is nothing 'degenerated' in the fact that one cannot infer the values of two variables from only one equation involving them. True degeneration would happen if in the formula for the coefficient c_2 the parameters b and f entered only as a ratio, e.g. as $(b/f)^2$. Clearly, it is not the case. We can therefore solve equations (106)-(107) separately for b and f. The result for f is

$$f(\Omega) = a_2 \frac{c_1}{c_2} = a_2 \beta^{-1} c_2^{-1}\,. \tag{109}$$

Unfortunately, the assumption of purely linear bias can be seriously questioned. Indeed, the value of the IRAS skewness is merely a half of the predicted one if $b = 1$ and this discrepancy is commonly attributed not to linear bias but to the fact that the IRAS survey systematically underestimates the density of galaxies in the cores of rich clusters (Bouchet et al. 1993). It simply means that the IRAS galaxies are nonlinearly (anti)biased tracers of mass distribution. In general, there is no a priori reason for the assumption of linear bias, justified for small (linear) fluctuations, to hold also in the case of fluctuations that are weakly nonlinear. Therefore, a correct method for estimating Ω from the comparison of the weakly nonlinear galaxy density with velocity fields should take into account the effects of nonlinear bias. Such a method has been invented by Bernardeau (private communication) and will be presented in the follow-up paper (Bernardeau, Chodorowski & Łokas, in preparation).

5 Summary and concluding remarks

In the present paper we derived a weakly nonlinear relation between cosmic density and velocity fields. In linear theory, the mass density contrast, δ, and the velocity divergence, $\nabla \cdot \mathbf{v}$, are in a given point linearly related. If the fields are nonlinear the density-velocity relation (DVR) is neither linear nor local. In weakly nonlinear regime, however, the spread

around the mean trend is so small that the conditional mean can serve as a very useful local approximation of a true nonlinear DVR.

We computed mean δ given $\theta := -f(\Omega)^{-1}H_0^{-1}\nabla \cdot \mathbf{v}$, that is $\langle\delta\rangle|_\theta$, up to third order in perturbation theory. According to our calculations, it is given by a third order polynomial in θ, $\langle\delta\rangle|_\theta = a_1\theta + a_2(\theta^2 - \sigma_\theta^2) + a_3\theta^3$. This formula constitutes therefore an extension of the formula $\langle\delta\rangle|_\theta = \theta + a_2(\theta^2 - \sigma_\theta^2)$, found by Bernardeau (1992), which involved second order perturbative solutions. Third order perturbative corrections not only introduce the cubic term but cause the a_1 coefficient to depart from unity as well, in contrast with the linear theory prediction. We computed the numerical values of the coefficients a_p for power-law spectra, as well as for standard CDM. The coefficients obey the hierarchy $a_3 \ll a_2 \ll a_1$, which means that the perturbative series converges very fast.

The key point of the method for disentangling Ω and bias lies in the fact that the coefficients a_p are practically Ω-independent. We have shown that the dependence of the coefficient a_2 on Ω is extremely weak. We have also given some arguments for the assumption that this is also the case for the coefficients a_1 and a_3. The detailed analysis of this problem will be given elsewhere.

Recently Ganon et al. (1996) have performed a set of N-body simulations for a CDM family of models in order to test different local approximations to DVR in weakly nonlinear regime. They tested, among others, an approximation of δ as a third order polynomial in θ. By visual inspection of the plots they provide one can see that the approximation works excellently for $|\delta|$ less than unity. It does not surprise us, since it is just the regime of applicability of perturbation theory. The values of the fitted parameters are in qualitative agreement with our perturbative calculations for a standard CDM: a_1 is slightly greater than unity, $a_2 \simeq 0.3$ and a_3 is equal to a few hundredths.

In order to derive accurately the values of a_p from N-body one should, however, treat properly the final velocity field, determined in a simulation only at a set of discrete points (final positions). A two-step smoothing procedure, commonly used, leads to rather substantial discrepancies between N-body simulations and analytical perturbative calculations of higher-order reduced moments (Łokas et al. 1995). Recently, Bernardeau & van de Weygaert (1996) proposed a new method for accurate velocity statistics estimation, based on the use of the Voronoi and Delaunay tessellations (adapted for a top-hat window function, however). The method proved to recover the tails of the velocity divergence distribution very accurately. Since the coefficients a_p are given by skewness and kurtosis-like quantities (see section 2), probing the tails of the density and velocity distribution, the application of the method is necessary to recover the accurate values of the coefficients from simulations. This will be the subject of the follow-up paper (Bernardeau, Chodorowski & Łokas, in preparation).

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Appendix A

All of the perturbative calculations in Section 3 are much simpler if they are performed in Fourier space. For the first order of the density contrast field we have

$$\delta_1(\mathbf{k},t) = D(t)\int \mathrm{d}^3x\ \delta_1(\mathbf{x})\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \tag{110}$$

and the inverse Fourier transform is

$$\delta_1(\mathbf{x},t) = D(t)(2\pi)^{-3}\int \mathrm{d}^3k\ \delta_1(\mathbf{k})\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{111}$$

For the calculations of the coefficients a_n the second and third order solutions for the density contrast and velocity divergence are needed and we give them here in the Fourier representation (e.g. Goroff et al. 1986). For the density field we have

$$\delta_2(\mathbf{k},t) = \frac{D^2}{(2\pi)^3}\int \mathrm{d}^3p\int \mathrm{d}^3q\ \delta_D(\mathbf{p}+\mathbf{q}-\mathbf{k})\delta_1(\mathbf{p})\delta_1(\mathbf{q})P_{2\delta}^{(s)}(\mathbf{p},\mathbf{q}) \tag{112}$$

$$\delta_3(\mathbf{k},t) = \frac{D^3}{(2\pi)^6}\int \mathrm{d}^3p\int \mathrm{d}^3q\int \mathrm{d}^3r\ \delta_D(\mathbf{p}+\mathbf{q}+\mathbf{r}-\mathbf{k})\delta_1(\mathbf{p})\delta_1(\mathbf{q})\delta_1(\mathbf{r})P_{3\delta}^{(s)}(\mathbf{p},\mathbf{q},\mathbf{r}). \tag{113}$$

The symmetrized kernels for the density field are of the form

$$P_{2\delta}^{(s)}(\mathbf{p},\mathbf{q}) = \frac{1}{14}J(\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{q}) \tag{114}$$

$$\begin{aligned} P_{3\delta}^{(s)}(\mathbf{p},\mathbf{q},\mathbf{r}) = \quad & A_\delta \quad [\,H(\mathbf{p}+\mathbf{q}+\mathbf{r},\mathbf{p})J(\mathbf{q}+\mathbf{r},\mathbf{q},\mathbf{r}) + \\ & + \quad H(\mathbf{p}+\mathbf{q}+\mathbf{r},\mathbf{q}+\mathbf{r})\ L(\mathbf{q}+\mathbf{r},\mathbf{q},\mathbf{r})] + \\ & +B_\delta \quad F(\mathbf{p}+\mathbf{q}+\mathbf{r},\mathbf{p},\mathbf{q}+\mathbf{r})L(\mathbf{q}+\mathbf{r},\mathbf{q},\mathbf{r}) + \\ & + \quad \begin{pmatrix} \mathbf{p}\to\mathbf{q} \\ \mathbf{q}\to\mathbf{r} \\ \mathbf{r}\to\mathbf{p} \end{pmatrix} + \begin{pmatrix} \mathbf{p}\to\mathbf{r} \\ \mathbf{q}\to\mathbf{p} \\ \mathbf{r}\to\mathbf{q} \end{pmatrix} \end{aligned} \tag{115}$$

where $A_\delta = 1/108$ and $B_\delta = 1/189$. In the expression above the notation follows that of Makino et al. (1992) i.e.

$$H(\mathbf{p},\mathbf{q}) = \frac{\mathbf{p}\cdot\mathbf{q}}{q^2} \tag{116}$$

$$F(\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{q}) = \frac{1}{2}\frac{|\mathbf{p}+\mathbf{q}|^2\mathbf{p}\cdot\mathbf{q}}{p^2q^2} \tag{117}$$

$$J(\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{q}) = 4\frac{(\mathbf{p}\cdot\mathbf{q})^{\mathbf{2}}}{p^2q^2} + 7\frac{p^2+q^2}{p^2q^2}\mathbf{p}\cdot\mathbf{q} + 10 \tag{118}$$

$$L(\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{q}) = 8\frac{(\mathbf{p}\cdot\mathbf{q})^{\mathbf{2}}}{p^2q^2} + 7\frac{p^2+q^2}{p^2q^2}\mathbf{p}\cdot\mathbf{q} + 6. \tag{119}$$

The solutions for the second and third order of the velocity divergence, θ_2 and θ_3, are of the same form as the density solutions except for the kernels $P_{2\delta}^{(s)}$ and $P_{3\delta}^{(s)}$ that must

be replaced by the corresponding kernels $P_{2\theta}^{(s)}$ and $P_{3\theta}^{(s)}$. The second order kernel for the velocity divergence is

$$P_{2\theta}^{(s)}(\mathbf{p},\mathbf{q}) = \frac{1}{14}L(\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{q}). \tag{120}$$

The third order kernel for the velocity divergence is obtained from the density kernel (115) by replacing the constants A_δ and B_δ with $A_\theta = 1/252$ and $B_\theta = 1/63$ respectively.

Appendix B

The expression $\langle\delta_1^2\delta_2\theta_2\rangle$ can be calculated in the same way as similar expressions $\langle\theta_1^2\theta_2^2\rangle$ or $\langle\delta_1^2\delta_2^2\rangle$. This is in fact an easier part of the kurtosis calculations as it involves only the second order perturbative solutions. We have

$$\begin{aligned}
\frac{\langle\delta_1^2\delta_2\theta_2\rangle}{\sigma^6} &= \frac{1}{196\pi^3\Gamma^3(\frac{n+3}{2})}\int \mathrm{d}^3p\int \mathrm{d}^3q\int \mathrm{d}^3r\ P(p)P(q)P(r)\times \\
&\times\ W(|\mathbf{p}+\mathbf{q}|)W(|\mathbf{r}-\mathbf{q}|)W(p)W(r)\times \\
&\times\ J(\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{q})L(\mathbf{r}-\mathbf{q},\mathbf{r},-\mathbf{q})
\end{aligned} \tag{121}$$

which after performing the integrations over angular variables becomes

$$\begin{aligned}
\frac{\langle\delta_1^2\delta_2\theta_2\rangle}{\sigma^6} &= \frac{8\pi}{\Gamma^3(\frac{n+3}{2})}\int \mathrm{d}p\int \mathrm{d}q\int \mathrm{d}r\ (pr)^{n+3/2}q^{n+1}\mathrm{e}^{-p^2-q^2-r^2}\times \\
&\times\ \left[\frac{34}{21}I_{\frac{1}{2}}(pq)-\left(\frac{p}{q}+\frac{q}{p}\right)I_{\frac{3}{2}}(pq)+\frac{8}{21}I_{\frac{5}{2}}(pq)\right]\times \\
&\times\ \left[\frac{26}{21}I_{\frac{1}{2}}(qr)-\left(\frac{q}{r}+\frac{r}{q}\right)I_{\frac{3}{2}}(qr)+\frac{16}{21}I_{\frac{5}{2}}(qr)\right].
\end{aligned} \tag{122}$$

Expanding the Bessel functions in powers

$$I_\nu(z) = \sum_{m=0}^{\infty}\frac{1}{m!\Gamma(\nu+m+1)}\left(\frac{z}{2}\right)^{\nu+2m} \tag{123}$$

and using the fact that

$$\int_0^\infty p^x\mathrm{e}^{-p^2}\mathrm{d}p = \frac{1}{2}\Gamma\left(\frac{1+x}{2}\right) \tag{124}$$

we obtain the result as a series of Gamma functions which can be summed numerically up to arbitrary accuracy. The numerical results are given in the third column of Table 5.

Appendix C

Using the second order solution for the density field (112) and corresponding one for the velocity divergence we obtain

$$\frac{\langle\alpha_2\beta_2\rangle}{\sigma^4} = \frac{D^4(t)}{98(2\pi)^6\sigma^4}\int \mathrm{d}^3p\int \mathrm{d}^3q\ P(p)P(q)W^2(|\mathbf{p}+\mathbf{q}|R)\ M(\mathbf{p}+\mathbf{q},\mathbf{p},\mathbf{q}) \tag{125}$$

where α and β stand for δ or θ. For different combinations we have

$$M = \begin{cases} J^2 & \text{for} \quad \alpha = \beta = \delta \\ J\ L & \text{for} \quad \alpha = \delta,\ \beta = \theta \\ L^2 & \text{for} \quad \alpha = \beta = \theta \end{cases} \tag{126}$$

with J given by equation (118) and L by equation (119).

The second type of terms involves the third order solution

$$\begin{aligned} \frac{\langle \alpha_1 \alpha_3 \rangle}{\sigma^4} &= \frac{6D^4(t)}{(2\pi)^6 \sigma^4} \int \mathrm{d}^3 p \int \mathrm{d}^3 q\ P(p) P(q) W^2(qR) \\ &\times \ \{\ A\ [\ H(\mathbf{q}, -\mathbf{p})\ J(\mathbf{p}+\mathbf{q}, \mathbf{p}, \mathbf{q}) \\ &\qquad\quad + H(\mathbf{q}, \mathbf{p}+\mathbf{q})\ L(\mathbf{p}+\mathbf{q}, \mathbf{p}, \mathbf{q})] \\ &\quad - \ B\ F(\mathbf{q}, -\mathbf{p}, \mathbf{p}+\mathbf{q})\ L(\mathbf{p}+\mathbf{q}, \mathbf{p}, \mathbf{q}) \} \,. \end{aligned} \tag{127}$$

If $\alpha = \delta$ the constants A_δ and B_δ must be used while if $\alpha = \theta$ they should be replaced respectively with A_θ and B_θ. The numerical values of the constants were given after equations (115) and (120) respectively.

After integration the expressions can be rewritten in a general way

$$\frac{\langle \alpha_i \beta_j \rangle}{\sigma^4} = \frac{D^4(t)}{2\pi^2 \sigma^4} \int_0^\infty \mathrm{d}k\ k^2\ W^2(kR) P_{ij}(k) \tag{128}$$

where

$$P_{22}(k) = \frac{k^3}{98(2\pi)^2} \int_0^\infty \mathrm{d}x P(kx) \int_{-1}^{+1} \mathrm{d}\mu P\left(k\sqrt{1+x^2-2x\mu}\right) \frac{f(x,\mu)}{(1+x^2-2x\mu)^2} \tag{129}$$

with

$$f(x,\mu) = \begin{cases} (3x + 7\mu - 10x\mu^2)^2 & \text{for} \quad \alpha = \beta = \delta \\ (3x + 7\mu - 10x\mu^2)(7\mu - x - 6x\mu^2) & \text{for} \quad \alpha = \delta,\ \beta = \theta \\ (7\mu - x - 6x\mu^2)^2 & \text{for} \quad \alpha = \beta = \theta \end{cases} \tag{130}$$

and

$$P_{13}(k) = \frac{k^3 P(k)}{(2\pi)^2} \int_0^\infty \mathrm{d}x P(kx) g(x) \tag{131}$$

with

$$g(x) = \begin{cases} \frac{1}{504}\left[\frac{12}{x^2} - 158 + 100x^2 - 42x^4 + \frac{3}{x^3}(x^2-1)^3(7x^2+2)\ln\frac{1+x}{|1-x|}\right] & \text{for} \quad \alpha = \beta = \delta \\ \frac{1}{168}\left[\frac{12}{x^2} - 82 + 4x^2 - 6x^4 + \frac{3}{x^3}(x^2-1)^3(x^2+2)\ln\frac{1+x}{|1-x|}\right] & \text{for} \quad \alpha = \beta = \theta. \end{cases} \tag{132}$$

In the case of power law spectra, all the expressions (128) diverge individually in the limit of $k \to 0$ and $kx \equiv q \to 0$ if $n \le -1$. Fortunately, as it is clear from the definition (94), in calculating Σ_2 similar expressions are subtracted and all such diverging terms cancel out. In the opposite limit of $k \to \infty$, $q \to \infty$, divergencies occur for the terms involving second order if $n > \frac{1}{2}$ and for the terms containing third order if $n > -1$. As thoroughly discussed

by Łokas et al. (1996) the only way to get along with those divergencies at $n > -1$ is to introduce a cutoff in the initial power spectrum at large wave-numbers. The results then depend on the cutoff wave-number k_c.

For integer values of the spectral index the integrals (128) can be performed by first finding the closed form expressions for P_{ij} similar to those proposed by Makino et al. (1992). In the limit of large cutoff wave-number $k_c \to \infty$ a simple analytical result can be found by identifying the terms that dominate P_{ij} in this limit and integrating term by term. Then the result for $n = -2$ does not depend on the cutoff.

ℓ	$H_\ell(\nu)$
0	1
1	ν
2	$\nu^2 - 1$
3	$\nu^3 - 3\nu$
4	$\nu^4 - 6\nu^2 + 3$
5	$\nu^5 - 10\nu^3 + 15\nu$
6	$\nu^6 - 15\nu^4 + 45\nu^2 - 15$

Table 1: The Hermite polynomials

spectral index n	a_2	a_3
-3.0	$\frac{4}{21} \approx 0.190$	$-\frac{40}{3969} \approx -0.0101$
-2.5	0.192	-0.00935
-2.0	0.196	-0.00548
-1.5	0.203	-0.000127
-1.0	0.213	0.00713
-0.5	0.227	0.0165
0	0.246	0.0279
0.5	0.270	0.0408
1.0	0.301	0.0532

Table 2: The coefficients a_2 and a_3 as functions of the spectral index n for scale-free power spectra and Gaussian smoothing

R	n_{eff}	$S_{3\delta}$	$S_{3\theta}$	a_2	$a_2(n_{eff})$
5	-0.946	3.459	2.177	0.2136	0.2141
10	-0.523	3.305	1.953	0.2253	0.2262
15	-0.255	3.227	1.821	0.2343	0.2355
20	-0.0654	3.179	1.729	0.2416	0.2430
50	0.462	3.082	1.482	0.2666	0.2680
100	0.722	3.051	1.359	0.2819	0.2830

Table 3: The comparison of the values of the coefficient a_2 for the CDM spectrum calculated using the exact (fifth column) and the approximate (sixth column) method

R	σ^2	a_1	a_2	a_3
5	0.578	1.196	0.214	0.00804
10	0.121	1.119	0.225	0.0160
15	0.0419	1.0755	0.234	0.0218
20	0.0185	1.0519	0.242	0.0263
50	0.000975	1.0125	0.267	0.0398
100	0.0000803	1.00357	0.282	0.0465

Table 4: The coefficients a_1, a_2 and a_3 for the CDM spectrum. In addition, the second column provides the values of the linear variance of the density (velocity divergence) field smoothed with a Gaussian filter for the CDM spectrum normalized as described in the text.

spectral index n	$\langle\theta_1^2\theta_2^2\rangle/\sigma^6$	$\langle\delta_1^2\delta_2\theta_2\rangle/\sigma^6$	$\langle\delta_1^3\delta_3\rangle/\sigma^6$	$\langle\theta_1^3\theta_3\rangle/\sigma^6$	Σ_4
-3.0	$\frac{1352}{441} \approx 3.07$	$\frac{1768}{441} \approx 4.01$	$\frac{682}{189} \approx 3.61$	$\frac{142}{63} \approx 2.25$	$\frac{5536}{1323} \approx 4.18$
-2.5	2.39	3.22	2.73	1.53	3.68
-2.0	1.87	2.61	2.11	1.01	3.31
-1.5	1.47	2.13	1.68	0.631	3.04
-1.0	1.16	1.76	1.38	0.332	2.84
-0.5	0.929	1.47	1.17	0.0799	2.71
0	0.755	1.24	1.02	-0.155	2.63
0.5	0.638	1.06	0.919	-0.398	2.58
1.0	0.584	0.916	0.855	-0.677	2.53

Table 5: The kurtosis-type quantities needed for the calculation of the coefficients a_1 and a_3 as functions of the spectral index n for scale-free power spectra and Gaussian smoothing

spectral index n	Σ_2	$\Delta S_3 S_{3\theta}/3 - \Sigma_4/2$	a_1
-2.0	0.369	-0.541	$1 - 0.172\ \sigma^2$
-1.9	0.399	-0.532	$1 - 0.134\ \sigma^2$
-1.8	0.440	-0.525	$1 - 0.0850\ \sigma^2$
-1.7	0.496	-0.518	$1 - 0.0217\ \sigma^2$
-1.6	0.576	-0.511	$1 + 0.0643\ \sigma^2$
-1.5	0.693	-0.506	$1 + 0.187\ \sigma^2$
-1.4	0.877	-0.501	$1 + 0.376\ \sigma^2$
-1.3	1.19	-0.497	$1 + 0.698\ \sigma^2$
-1.2	1.85	-0.493	$1 + 1.36\ \sigma^2$
-1.1	3.86	-0.490	$1 + 3.37\ \sigma^2$

Table 6: The contributions to the weakly nonlinear correction to the coefficient a_1 in the most interesting range of spectral indices for power law spectra and Gaussian smoothing

spectral index n	R	σ^2	a_1
-1.5	5	0.656	1.12
-1.4	5	0.638	1.24
-1.3	5	0.620	1.43
-1.5	12	0.176	1.03
-1.4	12	0.157	1.06
-1.3	12	0.140	1.10

Table 7: The values of the coefficient a_1 for the observationally preferred range of spectral indices of power law spectra and two different smoothing scales

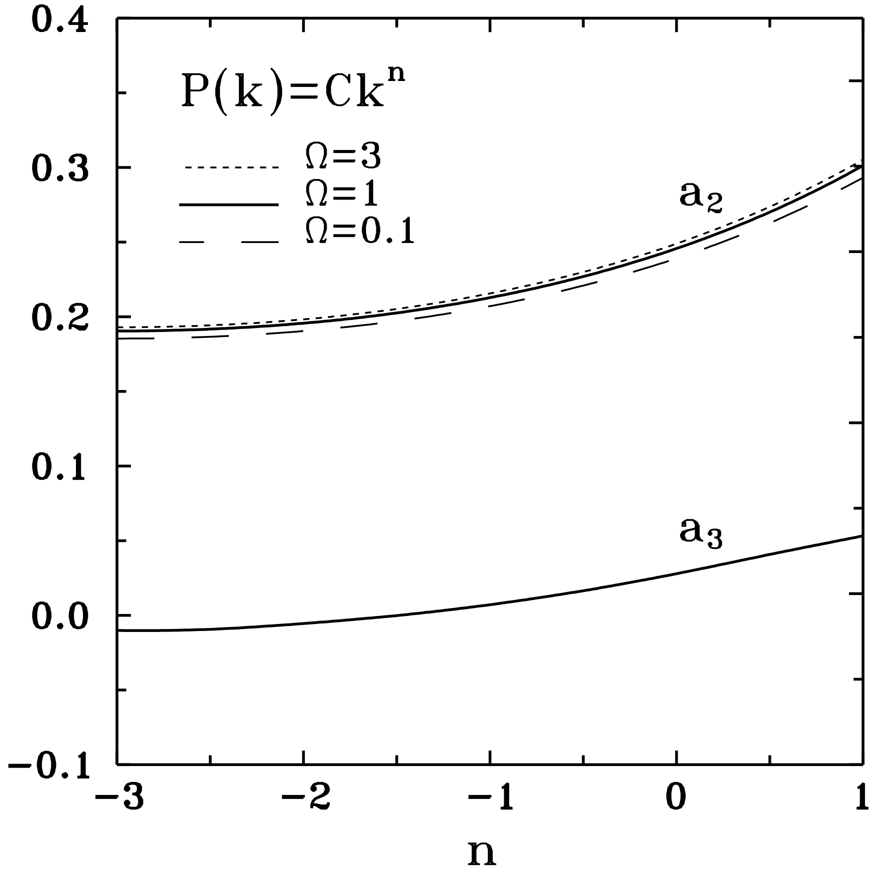

Figure 1: The coefficients a_2 and a_3 for scale-free power spectra and Gaussian smoothing as functions of the spectral index n. The solid lines correspond to the case of $\Omega = 1$. The coefficient a_2 is also shown for two other values of Ω parameter (dashed lines).

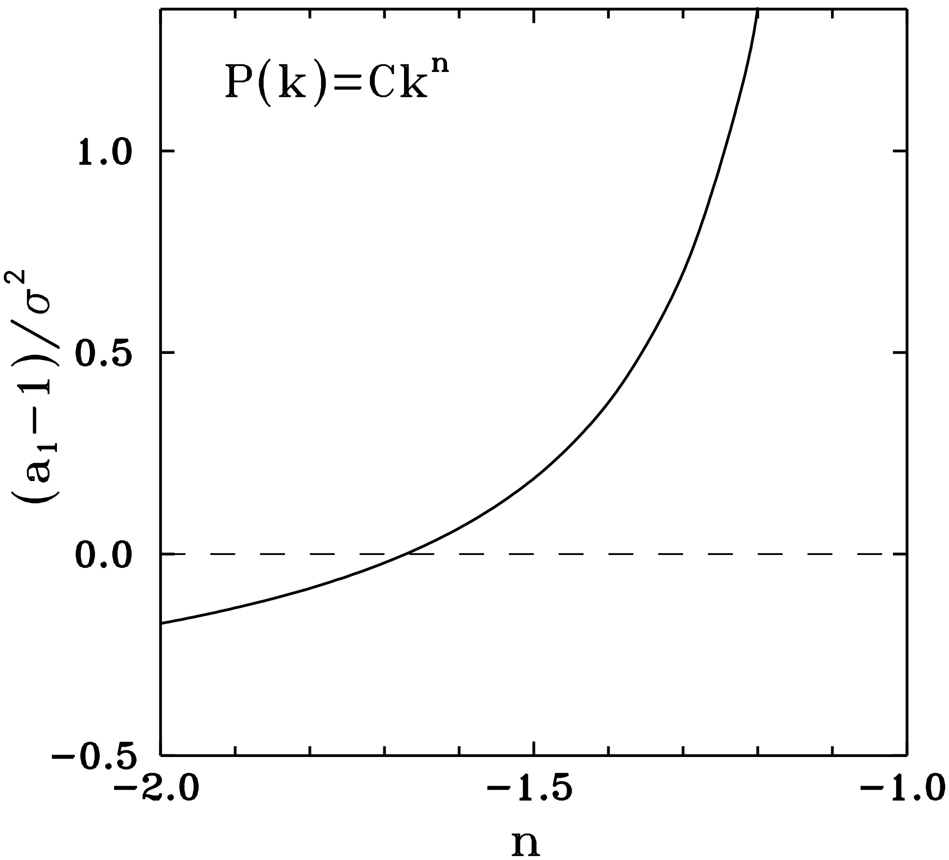

Figure 2: The weakly nonlinear correction to the coefficient a_1 divided by the linear variance σ^2 for scale-free power spectra and Gaussian smoothing in the observationally most interesting range of the spectral index n.

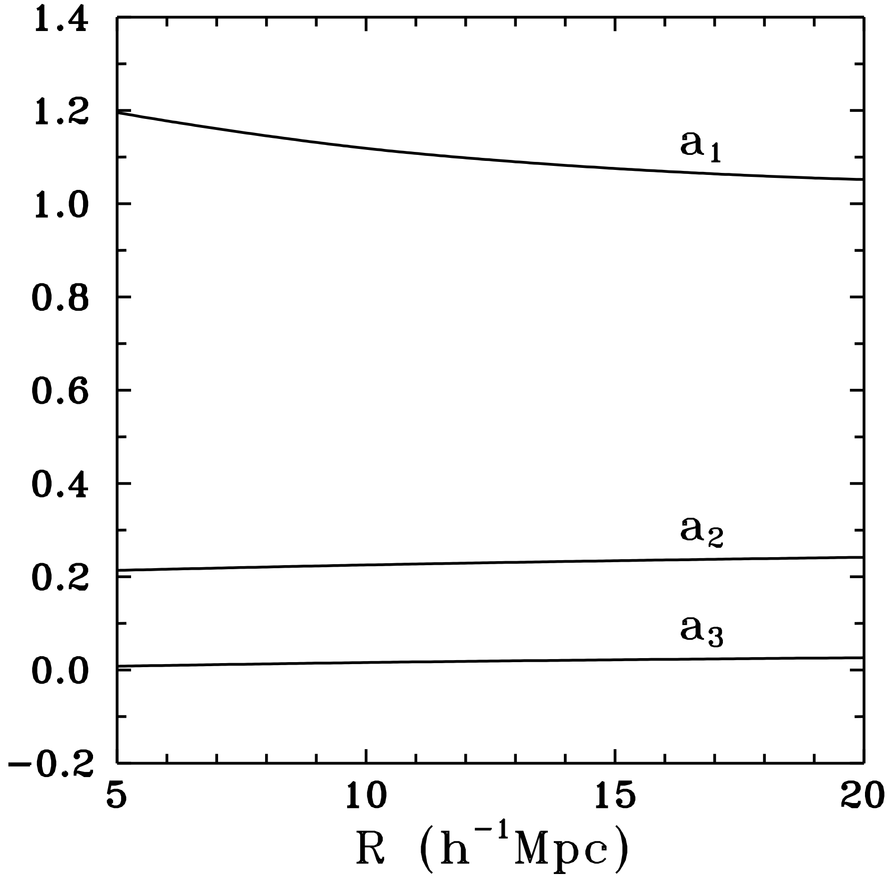

Figure 3: The coefficients a_1, a_2 and a_3 for the standard CDM spectrum normalized to (top-hat) $\sigma_8 = 1$ in the weakly nonlinear range of Gaussian smoothing scales.